

# Stability of Calabi flow near an extremal metric

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## Abstract

We prove that on a Kähler manifold admitting an extremal metric  $\omega$  and for any Kähler potential  $\varphi_0$  close to  $\omega$ , the Calabi flow starting at  $\varphi_0$  exists for all time and the modified Calabi flow starting at  $\varphi_0$  will always be close to  $\omega$ . Furthermore, when the initial data is invariant under the maximal compact subgroup of the identity component of the reduced automorphism group, the modified Calabi flow converges to an extremal metric near  $\omega$  exponentially fast.

## 1 Introduction

Let  $M$  be a Kähler manifold and  $\Omega$  be the Kähler class in  $H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$ . By the  $\partial\bar{\partial}$ -lemma, any Kähler metric  $\omega_\varphi$  in  $\Omega$  can be written as

$$\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$$

for some smooth real valued Kähler potential  $\varphi$ . The space of Kähler metrics is defined by

$$\mathcal{H} = \{\varphi \in C^\infty(M, \mathbb{R}) | \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}.$$

Donaldson [18], Mabuchi [21] and Semmes [22] independently defined a Weil-Peterson typed metric on  $\mathcal{H}$ , under which it becomes a non-positively curved infinite dimensional symmetric space. Chen [5] proved that any two points in  $\mathcal{H}$  can be connected by a  $C^{1,1}$  geodesic and that  $\mathcal{H}$  is a metric space, which verify two of the Donaldson's conjectures.

In order to tackle the existence of constant scalar curvature Kähler metric (cscK) problem, Calabi [2][3] introduced a well-known functional

$$Ca(\varphi) = \int_M S(\varphi)^2 \omega_\varphi^n,$$

where  $S(\varphi)$  is the scalar curvature of  $\omega_\varphi$ . The critical point of this Calabi functional is called an extremal Kähler metric. Calabi discovered that an extremal Kähler metric is a cscK if and only if the Calabi-Futaki invariant

is equal to zero. Later, he suggested that one may study the gradient flow of the K-energy to search for the cscK. This flow is defined as

$$\frac{\partial \varphi}{\partial t} = S - \underline{S} \quad (1)$$

and it decreases the Calabi energy. Since it is a fourth order equation, the maximal principle fails. In [19], Donaldson proposed a programme to study the convergence of the Calabi flow. On a Riemannian surface, P. Chruściel [17] proved that the flow exists for all time and converges to a cscK metric by using the Bondi mass. Later Chen [6] and Struwe [23] gave a different proof assuming the uniformization theorem. In Chen-Zhu [16], they removed the assumption of the uniformization theorem. For higher dimensions, the Calabi flow has been studied in Chen-He [8] [9] [10] and Tosatti-Weinkove [24]. In Chen-He [8], they proved that the Calabi flow can start from a  $C^{3,\alpha}$  Kähler potential and become smooth immediately as  $t > 0$ . One defines the little Hölder space  $c^{k,\alpha}$  to be the closure of smooth functions in the usual Hölder norm  $C^{k,\alpha}$ .

**Theorem 1.1.** *If  $\omega_{\varphi_0} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_0$  satisfies  $|\varphi_0|_{c^{3,\alpha}(M,g)} \leq K$ , and  $\lambda\omega < \omega_0 = \omega_{\varphi_0} < \Lambda\omega$ , where  $K, \lambda, \Lambda$  are positive constants, then the Calabi flow initiating from  $\varphi_0$  admits a unique solution*

$$\varphi(t) \in C([0, T], c^{3,\alpha}(M, g)) \cap C((0, T], c^{4,\alpha}(M, g))$$

for small  $T = T(\lambda, \Lambda, K, \omega)$ . More specifically, for any  $t \in (0, T]$ , there is a constant  $C = C(\lambda, \Lambda, K, \omega)$  such that

$$t^{1/4}(|\dot{\varphi}(t)|_{c^{0,\alpha}(M)} + |\varphi(t)|_{c^{4,\alpha}(M)}) \leq C, |\varphi(t)|_{c^{3,\alpha}(M)} \leq C.$$

**Remark 1.2.** *He [20] shows that the Calabi flow can start from  $\omega_\varphi$  where  $\varphi \in c^{2,\alpha}(M)$ .*

**Theorem 1.3.** *The solution obtained above belongs to  $C^0([0, T], c^{3,\alpha}(M)) \cap C^0((0, T], C^\infty(M))$ .*

Chen and He then use an energy argument to show that when Kähler manifolds admits a cscK  $\omega$  and the initial Kähler potential is  $C^{3,\alpha}$  small, the Calabi flow converges exponentially fast to a cscK nearby. In He [20], he improved this result for  $C^{2,\alpha}$  small initial Kähler potentials.

In this short note, we prove a parallel theorem for extremal Kähler metrics using a different method from Chen-Ding-Zheng [7]. In that paper, they defined a flow called the pseudo-Calabi flow. They proved the short time existence from  $c^{2,\alpha}$  initial Kähler potentials, the long time existence under uniform Ricci bounds and the stability near a cscK. Since the linearized operator of the pseudo-Calabi flow is not self-adjoint, they set up a unified frame to tackle the stability problem of the Kähler Ricci flow (c.f.

Chen-Zheng [15]), the Calabi flow and the pseudo-Calabi flow. This method strongly relied on the geometric structure of the space of Kähler metrics .

First, we will prove the long time existence of the Calabi flow.

**Theorem 1.4.** *On Kähler manifolds admitting an extremal metric  $\omega$  and for any positive constant  $K, \lambda$ , there is a small constant  $\epsilon$  depending on  $\omega, K, \lambda$ , such that for any Kähler potential  $\varphi_0$ , if*

$$|\varphi_0|_{C^{2,\alpha}(M)} < K, \quad \lambda\omega < \omega_{\varphi_0}, \quad \int_M |\varphi_0|^2 \omega^n < \epsilon,$$

*then the Calabi flow exists for all time.*

Next we want to study the modified Calabi flow. Let  $K$  be a maximal compact subgroup in the reduced automorphism group. Denote the corresponding Lie algebra of  $K$  by  $h_0(M)$ , which is the ideal of holomorphic vector fields with zeros. For any holomorphic vector field  $\tilde{Y} \in h_0(M)$ , denote  $\tilde{Y} = Y - \sqrt{-1}JY$ . Then there is a real function  $\theta_Y$  such that

$$L_{\tilde{Y}}\omega = L_Y\omega = \sqrt{-1}\partial\bar{\partial}\theta_Y(t)$$

and

$$\int_M \theta_Y \omega^n = 0.$$

For an arbitrary metric

$$\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi,$$

the corresponding  $\theta_{\tilde{Y}}(\varphi)$  is

$$\theta_{\tilde{Y}}(\varphi) = \theta_Y + \tilde{Y}(\varphi).$$

Following Futaki-Mabuchi [1], suppose  $\tilde{X}, \tilde{Y} \in h_0(M)$ , then the bilinear form

$$B(\tilde{X}, \tilde{Y}) = \int_M \theta_{\tilde{X}(\varphi)} \theta_{\tilde{Y}(\varphi)} \omega_\varphi^n$$

is independent of the choice of  $\omega_\varphi$  in the Kähler class  $\Omega = [\omega]$ .

Let  $\varphi(t)$  be a one parameter of Kähler potentials satisfying the Calabi flow equation and let  $\sigma(t)$  be the holomorphic group generated by  $X$ , the real part of the extremal vector field  $\tilde{X}$ . Then  $\sigma^*(t)\omega = \omega + i\partial\bar{\partial}\rho(t)$  satisfies the Calabi flow equation since

$$\frac{\partial\sigma^*(t)\omega}{\partial t} = L_X\omega(t) = i\partial\bar{\partial}(S(t) - \underline{S}).$$

Hence we can choose  $\rho(t)$  to be a parameter of Kähler potentials satisfying the Calabi flow equation starting from 0.

Let  $\psi(t) = \sigma(-t)^*(\varphi(t) - \rho(t))$ . Notice that, by definition,  $X = \sigma_*^{-1}(\frac{\partial}{\partial t}\sigma)$ . So we obtain the modified Calabi flow,

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= -X(\psi(t)) + \sigma(-t)^* \left( \frac{\partial \varphi(t)}{\partial t} - \frac{\partial \rho(t)}{\partial t} \right) \\ &= -X(\psi(t)) + \sigma(-t)^*(S(\varphi(t)) - \underline{S}) - (S(\omega) - \underline{S}) \\ &= S_\psi - \underline{S} - \theta_X - X(\psi) \\ &= S_\psi - \underline{S} - \theta_X(\psi) \end{aligned}$$

**Theorem 1.5.** *On Kähler manifolds admitting an extremal metric  $\omega$ , for any  $K$ -invariant Kähler potential  $\varphi_0$  close to  $\omega$  (in the sense of Theorem (1.4)), the modified Calabi flow exponentially converges to a nearby extremal metric.*

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## 2 Long time existence

First of all, we would like to give a rough estimate of the geodesic distance between any two Kähler potentials  $\varphi_0, \varphi_1$  when

$$\omega_{\varphi_0}, \omega_{\varphi_1} < \Lambda \omega.$$

**Lemma 2.1.**

$$d(\varphi_0, \varphi_1) < C(\Lambda) \left( \int_M |\varphi_0 - \varphi_1|^2 \omega^n \right)^{\frac{1}{2}}$$

*Proof.* Let  $\varphi_t = (1-t)\varphi_0 + t\varphi_1$  for  $0 \leq t \leq 1$ . Then

$$\begin{aligned} d(\varphi_0, \varphi_1) &\leq L(\gamma_t) \\ &= \int_0^1 \left( \int_M \left( \frac{\partial \gamma_t}{\partial t} \right)^2 \omega_{\gamma_t}^n \right)^{\frac{1}{2}} dt \\ &\leq \int_0^1 \left( \int_M (\varphi_0 - \varphi_1)^2 \omega_{\gamma_t}^n \right)^{\frac{1}{2}} dt \\ &\leq C(\Lambda) \left( \int_M |\varphi_0 - \varphi_1|^2 \omega^n \right)^{\frac{1}{2}}. \end{aligned}$$

□

We are ready to give a proof of Theorem 1.4.

*Proof.* Suppose that the conclusion fails, then there exist positive constants  $\mathcal{K}, \lambda, \Lambda$  and a sequence of  $\varphi_s^0$  such that

$$|\varphi_s^0|_{C^{2,\alpha}} < \mathcal{K}, \quad \lambda\omega < \omega_{\varphi_s^0} < \Lambda\omega, \quad \int_M |\varphi_s^0|^2 \omega^n < \frac{1}{s} \quad s = 1, 2, 3, \dots$$

By virtue of the short time existence theorem, we get a sequence of solutions  $\varphi_s(t)$  satisfying the flow equation (1) with  $\varphi_s(0) = \varphi_s^0$ . Let  $T_s$  be the first time such that

$$|\varphi_s(T_s) - \rho(T_s)|_{C^{2,\alpha}(\omega(T_s))} = 2C \quad \text{holds}$$

or

$$\lambda\omega(T_s) < \omega_{\varphi_s(T_s)} < \Lambda\omega(T_s) \quad \text{fails,}$$

where the constant  $C$  is from Theorem 1.1. Then  $T_s$  is bounded from below for sufficiently large  $s$ . Otherwise there is a subsequence of  $\varphi_s(T_s)$  converging to  $\varphi_\infty$  in the  $C^{2,\alpha'}(\omega)$  sense, where  $\alpha' < \alpha$ . Notice that  $\lambda\omega \leq \omega_{\varphi_\infty} \leq \Lambda\omega$ , but that  $\lambda\omega < \omega_{\varphi_\infty} < \Lambda\omega$  fails.

On the other hand, Lemma (2.1) shows that  $d(0, \varphi_s^0) \rightarrow 0$  as  $s \rightarrow \infty$ . Since the distance function decreases under the Calabi flow, we have

$$d(\rho(T_s), \varphi_s(T_s)) \rightarrow 0$$

as  $s \rightarrow \infty$ . Let  $\varphi_\infty(t)$  be one parameter potentials satisfying the Calabi flow equation initiating from  $\varphi_\infty$ . Then for  $t_0 \geq t$ ,

$$\begin{aligned} d(\rho(t_0), \varphi_\infty(t_0)) &\leq d(\rho(t), \varphi_\infty(t)) \\ &\leq d(\rho(t), \rho(T_s)) + d(\rho(T_s), \varphi_s(T_s)) + d(\varphi_s(T_s), \varphi_\infty(t)). \end{aligned}$$

By Lemma (2.1),  $d(\varphi_s(T_s), \varphi_\infty(t)) \rightarrow 0$  as  $s \rightarrow \infty$  and  $t \rightarrow 0$ . Hence  $\rho(t_0) = \varphi_\infty(t_0)$ , which implies  $0 = \varphi_\infty$ , a contradiction.

Moreover, from Theorem 1.3, we obtain the higher order uniform bounds of the sequence of the solutions:

$$|\varphi_s(T_s) - \rho(T_s)|_{C^{k,\alpha}(\omega(T_s))} \leq C(k), \quad \forall k \geq 0.$$

Therefore we can choose a subsequence of  $\phi_s = \sigma(-T_s)^*(\varphi_s(T_s) - \rho(T_s))$  so that

$$\phi_s \rightarrow \phi_\infty \text{ in } C^{k,\alpha}(\omega), \forall k \geq 0,$$

and

$$|\phi_\infty|_{C^{2,\alpha}(\omega)} = 2C \quad (\text{or } \lambda\omega < \omega_{\phi_\infty} < \Lambda\omega \text{ fails}).$$

However, this contradicts the fact that  $d(0, \phi_\infty) = 0$ . □

**Corollary 2.2.** *Given a Kähler potential  $\varphi_0$  close to 0 in the sense of Theorem 1.4, then the modified Calabi flow stays in a neighborhood of 0. If  $\varphi_0$  is  $\mathcal{K}$ -invariant, then the modified Calabi flow converges to an extremal metric nearby.*

*Proof.* That the modified Calabi flow stays in a neighborhood of 0 can be easily seen from the regularity Theorem 1.3. Notice that the Calabi flow decreases the Calabi energy, i.e.

$$\frac{\partial}{\partial t} Ca(\omega_\varphi) = -2 \int_M \mathcal{L}_\varphi(S_\varphi) S_\varphi \omega_\varphi^n,$$

where  $\mathcal{L}_\varphi$  is the Lichnerowicz operator with respect to  $\omega_\varphi$ . It follows that we can take a sequence of  $t_j \rightarrow \infty$  such that

$$\lim_{j \rightarrow \infty} \int_M \mathcal{L}_{\varphi(t_j)}(S_{\varphi(t_j)}) S_{\varphi(t_j)} \omega_{\varphi(t_j)}^n = 0.$$

Then there is a subsequence of  $t_j$  such that  $\psi(t_j)$  converges to a potential  $\psi_\infty$  in  $C^\infty$  and

$$\int_M \mathcal{L}_{\psi_\infty}(S_{\psi_\infty}) S_{\psi_\infty} \omega_{\psi_\infty}^n = 0.$$

Hence  $\omega_{\psi_\infty}$  is an extremal metric. If  $\varphi_0$  is  $\mathcal{K}$ -invariant, then  $\psi_\infty$  is a fixed point under the modified Calabi flow and the modified Calabi flow decreases the geodesic distance between  $\psi(t)$  and  $\psi_\infty$ . Hence the flow converges to  $\psi_\infty$ .  $\square$

### 3 Exponential decay

We define the modified Calabi energy as

$$\widetilde{Ca}(\psi) = \int_M (S(\psi) - \underline{S} - \theta_X(\psi))^2 \omega_\psi^n.$$

The evolution of the modified Calabi energy along the modified Calabi flow is

$$\begin{aligned} \partial_t \int_M \psi^2 \omega_\psi^n &= \int_M (2\dot{\psi}\ddot{\psi} + \dot{\psi}^2 \Delta_\psi \psi) \omega_\psi^n \\ &= 2 \int_M \dot{\psi}(\dot{S}_\psi - \dot{\theta}_X(t) - \dot{\psi}_i \dot{\psi}^i) \omega_\psi^n \\ &= 2 \int_M \dot{\psi}(-L\dot{\psi} + \dot{\psi}^i S_i - \dot{\theta}_X(t) - \dot{\psi}_i \dot{\psi}^i) \omega_\psi^n \\ &= 2 \int_M \dot{\psi}(-L\dot{\psi} + \dot{\psi}^i (\dot{\psi}_i + \theta_X(t)_i) - \dot{\theta}_X(t) - \dot{\psi}_i \dot{\psi}^i) \omega_\psi^n \\ &= 2 \int_M \dot{\psi}(-L\dot{\psi} + \dot{\psi}^i \dot{\psi}_i + X(\dot{\psi}) - X(\dot{\psi}) - \dot{\psi}_i \dot{\psi}^i) \omega_\psi^n \\ &= -2 \int_M \dot{\psi} L \dot{\psi} \omega_\psi^n. \end{aligned}$$

In this computation we use the identities

$$\dot{\psi}^i(\theta_{X_i} + (X(\psi))_i) = \dot{\psi}^i(\theta_X(t))_i = X(\dot{\psi}).$$

The modified Calabi-Futaki invariant

$$\tilde{F}(\tilde{Y}) = F(\tilde{Y}) - B(\tilde{X}, \tilde{Y}) = \int_M \theta_{\tilde{Y}}(\psi)(S - \underline{S} - \theta_{\tilde{X}}(\psi))\omega_{\psi}^n \quad (2)$$

is equal to zero when  $M$  admits an extremal metric  $\omega$ . This shows that the direction of the modified Calabi flow is always perpendicular to the kernel of the Lichnerowicz operator. To obtain exponential convergence, one needs to give a uniform lower bound of the the first eigenvalue of  $L_t$  along the modified Calabi flow. More precisely, we have the following lemma which is similar to Chen-Li-Wang [11].

**Lemma 3.1.** *Along the modified Calabi flow, there is a positive constant  $\lambda > 0$  such that for sufficiently large  $t$  and for any*

$$f \in A_t = \{f \in C_R^\infty(M) \mid \int_M f \omega_{\psi(t)}^n = 0 \text{ and } \int_M \theta_Y(t) f \omega_{\psi(t)}^n = 0, \forall \tilde{Y} \in h_0(M)\},$$

we have

$$\int_M L_t(f) f \omega_{\psi(t)}^n \geq \lambda \int_M f^2 \omega_{\psi(t)}^n.$$

*Proof.* If not, there must be a sequence  $\psi_s = \psi(s)$  and  $f_s$  such that

$$\int_M |(f_s)_{ij}|^2 \omega_{\psi_s}^n < \frac{1}{s}; \int_M f_s^2 \omega_{\psi_s}^n = 1; \int_M f_s \omega_{\psi_s}^n = 0. \quad (3)$$

Since the  $C^l$  norm of  $\psi_s$  is uniformly bounded for any  $l \geq 0$ . Using the Ricci identity

$$\int_M |(f_s)_{i\bar{j}}|^2 \omega_{\psi_s}^n = \int_M |(f_s)_{ij}|^2 \omega_{\psi_s}^n + \int_M R^{i\bar{j}}(f_s)_i (f_s)_{\bar{j}} \omega_{\psi_s}^n$$

and the interpolation inequality, we conclude that  $f_s$  are uniformly  $W^{2,2}$  bounded. So we can pass to the limit and get

$$\int_M |(f_\infty)_{ij}|^2 \omega_{\psi_\infty}^n = 0; \int_M f_\infty^2 \omega_{\psi_\infty}^n = 1; \int_M f_\infty \omega_{\psi_\infty}^n = 0. \quad (4)$$

Since in local coordinates,  $\uparrow \bar{\partial} f_\infty$  is holomorphic in the weak sense,  $f_\infty$  is smooth indeed. From the assumption of  $A_t$  we have

$$\int_M \theta_Y(\psi_\infty) f_\infty \omega_{\psi_\infty}^n = 0, \forall \tilde{Y} \in h_0(M).$$

In particular, we may choose  $\tilde{Y} = \uparrow \bar{\partial} f_\infty \in h_0(M)$ . Hence,

$$\int_M f_\infty^2 \omega_{\psi_\infty}^n = 0.$$

This contradicts (4). □

It is easy to see that  $\widetilde{Ca}(\psi(t)) \leq Ce^{-\lambda t}$ . To get exponential convergence of  $\psi(t)$ , we calculate the evolution formula for  $\int_M |\nabla^k(\psi(t) - \psi_\infty)|^2 \omega^n$ :

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_M |\nabla^k(\psi(t) - \psi_\infty)|^2 \omega^n \\
&= \int_M \nabla^k(S - \underline{S} - \theta_X(\psi)) * \nabla^k(\psi(t) - \psi_\infty) \omega^n \\
&= \int_M (S - \underline{S} - \theta_X(\psi)) * \nabla^{2k}(\psi(t) - \psi_\infty) \omega^n \\
&\leq \left( \int_M (S - \underline{S} - \theta(X))^2 \omega^n \right)^{1/2} \left( \int_M |\nabla^{2k}(\psi(t) - \psi_\infty)|^2 \omega^n \right)^{1/2} \\
&\leq C \|S - \underline{S} - \theta(X)\|_{L^2(\omega)} \\
&\leq C \|S - \underline{S} - \theta(X)\|_{L^2(\omega_t)} \\
&\leq Ce^{-\lambda_2 t}.
\end{aligned}$$

By the Sobolev embedding, we conclude that

$$\|\psi_t - \psi_\infty\|_{C^l(\omega)} \leq \|\psi_t - \psi_\infty\|_{W^{k,2}(\omega)} \leq Ce^{-\lambda_2 t}.$$

Hence we obtain the result stated in Theorem 1.5.

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